ON TESTING OF GENERAL RANDOM CLOSED SET MODEL HYPOTHESIS

T. MRKVIČKA

Mathematical Department, University of South Bohemia, Jeronýmova 10, 37115 České Budějovice, Czech Republic, E-mail: mrkvicka@pf.jcu.cz

Abstract: A new method of testing the random closed set model hypothesis (for example: the Boolean model hypothesis) for a stationary random closed set $\Xi \subseteq \mathbb{R}^d$ with values in the extended convex ring is introduced. The method is based on the estimating of the intrinsic volumes densities of the ε-parallel sets to Ξ and comparing them with its envelopes produced from simulations of the model given by the tested hypothesis. The power of this test is estimated for planar Boolean model hypothesis and two different alternatives and the resulted powers are compared to the powers of known Boolean model tests. The method is applied on the real data set of the heather incidence.

Keywords: Boolean model, Boolean model hypothesis, contact distribution function, Euler-Poincaré characteristic, Intrinsic volumes, Laslett's transform

Mathematics Subject Classification: 60D05, 62G05.

1. Introduction

Simulation based tests are wide spread in stochastic geometry for testing a model hypothesis. For example, the simulation based tests of random point process model hypothesis are described in Dam et al. (1999) or Moller & Waagepetersen (2004). When testing a random closed set model hypothesis (for example a Boolean model), it is important to choose a characteristic of the model which is able to distinguish between different models.

The characteristics which are commonly used for describing a closed set are intrinsic volumes. The intrinsic volumes $V_0(K),\dots,V_d(K)$ of a convex body $K\subseteq\mathbb{R}^d$ are determined by the Steiner formula

$$V_d(K_\varepsilon)=\sum_{i=0}^{d}\varepsilon^i\omega_i V_{d-i}(K),$$

where V_d is the volume (d-dimensional Lebesgue measure), $K_\varepsilon=\{x\in\mathbb{R}^d : \mathrm{dist}\,(x,K)\le\varepsilon\}$ the (closed) ε-parallel set to K and ω_k denotes the volume of the unit ball in $\mathbb{R}^k$. (Under a different normalization, they are known as quermassintegrals or Minkowski functionals.) The intrinsic volumes can be extended additively to polyconvex sets (sets from the convex ring). For details see Schneider (1993).

We consider a stationary random closed set Ξ in $\mathbb{R}^d$ with values in the extended convex ring (e.g. Ξ can be represented as a locally finite union of convex bodies). Under certain integrability condition, the *intrinsic volume densities* of Ξ can be defined as in Schneider & Weil (2000) or Rataj (2005) by

$$\overline{V}_k(\Xi)=\lim_{r\to\infty}\frac{\mathbb{E}V_k(\Xi\cap rB)}{V_d(rB)},$$

where B is any convex body with nonempty interior; for a detailed introduction see Schneider & Weil (2000) or Stoyan at al. (1995). In the plane, $\overline{V}_2(\Xi)$ is the volume density, $\overline{V}_1(\Xi)$ is one half of the circumference density of $\partial\Xi$ and $\overline{V}_0(\Xi)$ is the mean Euler number density.

The intrinsic volumes densities are not able to distinguish different models alone, therefore we use in our test the intrinsic volumes densities of various (closed) ε-parallel sets to Ξ. This can easily reflect the differences between regular - Boolean - cluster models.

There exist several methods for estimating intrinsic volumes densities Ohser & Mücklich (2000), Nagel at al. (2000), Rataj (2004), Schmidt & Spodarev (2004) and Mrkvička & Rataj (2006). The last method is especially suitable for purposes of estimating the intrinsic volumes densities of ε-parallel sets due to the following two reasons:
1) It is an unbiased estimator of $V_k(\Xi_\varepsilon)$, $k=0,\dots,d-1$ for $\varepsilon>\delta$,

where δ depends on the estimating procedure.
2) When we estimate $V_k(\Xi),\ k = 0, \dots, d-1$ using method described in Mrkvička & Rataj (2006), we use the approximation by estimating $V_k(\Xi_\varepsilon),\ k = 0, \dots, d-1$ for as small $\varepsilon > 0$ as it is possible. But we are limited by the resolution of a discretized image, therefore the estimator of $V_k(\Xi)$ is biased and it is shown that this bias is crucial for the uncertainty of the estimator. Despite such a bias the estimator described in Mrkvička & Rataj (2006) is comparable with any other estimator, see Mrkvička & Rataj (2006). Therefore, when the crucial bias is removed (due the the estimation of $V_k(\Xi_\varepsilon),\ k = 0, \dots, d-1$), this estimator will have the smallest uncertainty from all available estimators.

The chosen estimator of intrinsic volumes densities estimates only $V_k(\Xi_\varepsilon),\ k = 0, \dots, d-1$ therefore the classical point-counting estimator is used for estimating volume density $V_d(\Xi_\varepsilon)$.

Concerning the organization of the work, we give a short description of the available tests in Section 2. The proposed test is described in Section 3. Then we compare the powers of all tests with respect to the certain alternatives in Section 4. The real data study is performed in Section 5 on the heather incidence data which were studied first in Diggle (1981). Here a proposed test is applied on the data set with the hypothesis that the data fits the Boolean model of disks with the parameters estimated in Diggle (1981). This hypothesis was rejected which confirms the conjecture given in Diggle (1981), that the data set does not fit the Boolean model. Thus another model is then proposed which is not rejected by the proposed test.

2. Tests description

In this section we give a summary of the tests available for testing the Boolean model assumption and the general model assumption. For testing the Boolean model assumption one can use the following two procedures.

2.1. **Laslett's test.** For details, see Cressie (1997) or Molchanov (1997).

Briefly:

(1) Take the tangent points in a certain direction u.
(2) Apply Laslett's transform on those points.
(3) Test the Poisson property of the transformed points. If it is not Poisson then the Boolean model hypothesis is rejected.

Disadvantages: A considerably big part of the tangent points have to be omitted because of the dependencies between transformed points and the observation window. Clustering or regularity may be lost after the Laslett's transform, see Cressie (1997), p. 769.

For testing the Poisson property we used, in the simulation study, the method using the second-order moment function Stoyan et al. (1995), p. 50-51.

2.2. **Graphical test for contact distribution function.** For details, see Stoyan et al. (1995) or Molchanov (1997).

Usually the linear contact distribution function $\hat{H}_l(r)$ and the quadratic contact distribution function $\hat{H}_q(r)$ are estimated. Then the normalized logarithm $\hat{H}_l^l(r) = -\frac{1}{r}\ln(1 - \hat{H}_l(r))$ is graphically compared with a constant function. And the normalized logarithm $\hat{H}_q^l(r) = -\frac{1}{r}\ln(1-\hat{H}_q(r))$ is graphically compared with a linear function. If both are graphically satisfied then the Boolean model is accepted.

Disadvantages: No significance level is available.

For the purposes of our simulation study and the comparison of the tests we need a clear decision algorithm whether the Boolean model is rejected or not. Therefore we calculated the standard deviation of $\hat{H}_l^l(r), r = 1, \ldots, 24$ (SD_L) and R^2 of $\hat{H}_q^l(r), r = 1, \ldots, 24$ with respect to linear regression model (R_Q^2) for the data. Then we performed 39 simulations of the Boolean model with parameters estimated using estimated quadratic contact distribution function Molchanov (1997), p. 77-79 and we calculated SD_L and R_Q^2 for each simulation. We reject the Boolean model hypothesis if SD_L computed from the data is maximal or if R_Q^2 computed from the data is minimal. Under the null hypothesis and assumption of the independence of SD_L and R_Q^2 this test has the significance level 0.0494.

For testing the general model assumption one can use the following procedure.

2.3. **Simulation-based tests.** For details see Dam et al. (1999), p. 48-50. Any simulation based test is applicable on any model assumption where we are able to estimate the model parameters. Generally, when a simulation based test is performed, the following procedure is done:

(1) Summary statistic of the random process is chosen and it is estimated from the data in n different points.
(2) The parameters of the assumed model are estimated.
(3) N independent samples of the model with estimated parameters are made and the envelopes of the chosen summary statistics are produced.
(4) We reject the hypothesis if there are more than k observations of the all summary statistics outside of the envelopes. Here k can be set in the way that the probability $P_{N,n,k}$ (i.e. the probability that k or more observations fall outside of the envelopes, under the assumption of the hypothesis and the independence between individual observations) is approximately equal to 0.05.

We found only one application of this kind of the test in the literature, see Diggle (1981). There the quadratic contact distribution function is used as a summary statistic.

In our simulation study we processed this test in the following way.

(1) The quadratic contact distribution function is estimated from the data in 24 different points (The points are same as in the proposed test).
(2) The parameters of the assumed model, which is the Boolean model in our simulation study, are estimated using estimated quadratic contact distribution function Molchanov (1997), p. 77-79.

(3) The envelope is made of 39 independent samples.

(4) We reject the hypothesis if there are 4 or more observation of the summary statistic outside of the envelope. ($P_{39,24,4} = 0.0298$)

3. Proposed test

We propose a simulation based test for testing the general model assumption in case we are able to estimate the parameters of the model. We propose the following procedure:

(1) We chose intrinsic volumes densities $V_k(\Xi_{\varepsilon_i}),\ k = 0, \ldots, d,\ i = 1, \ldots, 24$ of the ε_i-parallel sets as summary statistics. The discretized version of the parallel set Ξ_{ε_i} is produced as dilation of the set Ξ by a discretized disk with a radius ε_i. The radii of the disks were determined as in Mrkvička & Rataj (2006), Section 6. We chose 24 different disks with radii $\varepsilon_1, \ldots, \varepsilon_{24}$ evenly spread between 1 and 25 pixels of the image. Here we can see that the test is dependent on the resolution of the original image and the size of the particles, but the radii can be shrunk or spread to cover the interesting area of pixels. When the envelopes are made from the simulations, the estimated intrinsic volumes densities $V_k(\Xi_{\varepsilon_1})$ vary a lot for different simulations thus the envelopes are wide. Therefore we fixed first point by choosing a normalized summary statistics:

$$\frac{V_k(\Xi_{\varepsilon_i})}{V_k(\Xi_{\varepsilon_1})}, \quad i = 2, \ldots, 24, \quad k = 0, \ldots, d.$$

This normalized summary statistics reflect the interactions between the particles and omits the number and size of the particles. The estimation of $V_d(\Xi_{\varepsilon_i})$ is performed by the standard unbiased point counting estimator and the estimation of $V_0(\Xi_{\varepsilon_i}), \ldots, V_{d-1}(\Xi_{\varepsilon_i})$ is performed by the unbiased estimator described in Mrkvička & Rataj (2006).

(2) In our simulation study we tested the Boolean model assumption. The parameters of the Boolean model are estimated using

empirical intrinsic volume densities $V_0(\Xi),\dots,V_d(\Xi)$ Molchanov (1997), p. 81-83. Using the same summary statistics for estimating the parameters and model validation leads to a greater acceptance of the model on one side but on the other side it eliminates other influences than the position of the particles. Therefore we chose the same summary statistics. The intrinsic volume densities $V_0(\Xi),\dots,V_{d-1}(\Xi)$ are approximated by $V_0(\Xi_{\varepsilon_1}),\dots,V_{d-1}(\Xi_{\varepsilon_1})$ as it is described in Mrkvička & Rataj (2006).

(3) The envelopes were made of 99 independent samples.

(4) We reject the hypothesis if there are 4 or more observation of the summary statistics outside of the envelopes. ($P_{99,69,4} = 0.0497$)

4. Simulation study

First we estimated the powers of all previously described tests with respect to certain alternatives. We chose two alternatives (cluster model and regular model). In both cases the model was tested on the Boolean model hypothesis.

4.1. **Cluster model.** As a representative of the cluster model we chose the germ-grain model in $\mathbb{R}^2$ where the germs form a Mattérn cluster point process with the intensity $\lambda = 0.0012$. The cluster point process is constructed in two steps. First the Poisson point process of cluster centers is generated with the intensity $\alpha = \lambda/\nu$, where ν is the mean number of points per cluster. Then, in each cluster, there are generated points which number follows the Poisson distribution with parameter $\nu = 6$ and which are uniformly distributed in the disk with radius $R = 50$ pixels around the cluster center. The germs are disks with radii which follow the lognormal distribution with parameters $[\mu = 2.690, \sigma = 0.19]$. The point process of germs is the process of centers of disks. We made 30 simulations in the observation window $W = 500 \times 500$ pixels. The typical observation of this model is shown on Figure 1.

Figure 1

For each simulation we performed 4 tests.

(1) Laslett's test.
(2) Test on the base of the linear and quadratic contact distribution function.
(3) Simulation-based test with summary statistic - quadratic contact distribution function.
(4) Proposed test: Simulation-based test with summary statistics - intrinsic volume densities of the parallel sets.

For the test 1 no assumption is needed. For the simulation based tests and test 2 we have to assume shape of the particles and distribution of the size of the particles. Wrong assumption can lead to a lower acceptance of the model thus we chose correct assumptions because then we can expect lower number of successive rejections of the Boolean model hypothesis in tests 2, 3, 4. Thus only parameters (λ, μ, σ) are necessary to estimate. Both approaches for model parameters estimation (using quadratic contact distribution function for tests 2, 3 and using intrinsic volume densities for test 4) lead to estimates of $(\lambda, \bar{A}, \bar{U})$, where $\bar{A} = \mathbb{E}A(\Xi_0)$ is the mean area of the typical grain Ξ_0 and $\bar{U} = \mathbb{E}U(\Xi_0)$ is the mean perimeter of the typical grain Ξ_0. From estimates of $(\lambda, \bar{A}, \bar{U})$ we estimated (λ, μ, σ) by a classical moment method.

The numbers of successive rejections of the Boolean model hypothesis are summarized in Table 1.

Table 1

For comparison of the tests 3 and 4 we proceed the paired asymptotic test with the resulted p-level 0.016. Thus we reject the hypothesis that both tests have the same power against the given cluster model alternative.

4.2. **Regular model.** As a representative of the regular model we chose the germ-grain model in $\mathbb{R}^2$ where the germs form a regular point process with the intensity $\lambda = 0.00136$. The regular point process is constructed from evenly scattered points in $\mathbb{R}^2$ when each point is then shifted in random direction by a distance h. The distance h was chosen to have the uniform distribution with parameters $[0, 10]$ pixels. The germs are disks with radii which follow the lognormal distribution with parameters $[\mu = 2.6903, \sigma = 0.19]$. The point process of germs is the process of centers of disks. We made 30 simulations in the observation window $W = 500 \times 500$ pixels. The typical observation of this model is shown on Figure 1. We chose big volume fraction for the regular model because for the small volume fraction the best way to test the hypothesis would be detecting the disk centers and using the methods for point processes. Since there are small spaces between particles of the simulated data we used, for the dilation Ξ_ε, the discretized disks with radii $\varepsilon_1, \dots \varepsilon_{24}$ evenly spread between 1 and 12 pixels of the image. (Greater disks produce after dilation just a white rectangle.)

For each simulation we performed same tests and same parameter estimation procedure as for the regular model. The numbers of successive rejections of the Boolean model hypothesis are summarized in Table 1.

For the comparison of the tests 3 and 4 we proceed the paired asymptotic test with the resulted p-level 0.086. Thus we do not reject the hypothesis that both tests have the same power against the given regular model alternative. Moreover we used parameters of the Boolean model estimated using intrinsic volumes for the tests 2 and 3 (instead of parameters estimated using contact distribution function) because the estimation which use contact distribution function did not worked for this example. This probably caused a slight increase of the successive rejection for the tests 2 and 3.

4.3. **Sensitivity of the proposed test to the wrong assumptions.** In this subsection, we will look at the sensitivity of the proposed test to

(1) the wrong choice of the shape of the particles,
(2) the wrong choice of the distribution of the particle sizes,

(3) the wrong choice of the particle shapes and distribution of the particle sizes.

More specifically:

(1) We simulated 100 Boolean models with ellipse grains where the length a of the shorter axis of the ellipse had lognormal distribution with parameters $[\mu = 3.11, \sigma = 0.31]$ and the length of the longer axis of the ellipse was $b = aU$, where U is a random variable with uniform distribution with parameters $[1, 2]$. We tested the hypothesis that the model is Boolean with disk grains, where the radii of the disks have lognormal distribution.
(2) We simulated 100 Boolean models with disk grains where the radii of the disks had uniform distribution with parameters $[20, 40]$ pixels. We tested the hypothesis that the model is Boolean with disk grains, where the radii of the disks have lognormal distribution.
(3) The same situation as in (1) but the length a of the shorter axis of the ellipse had uniform distribution with parameters $[10, 37]$.

The observation window was 500×500 pixels and the intensity was 0.00018 in all three cases. The parameters of the simulated models were set in such a way that their typical primary grains have same mean volume and mean circumference as the model in the tested hypothesis (Boolean model with disk grains, where the radii of the disks have lognormal distribution with parameters $[\mu = 3.383, \sigma = 0.19]$). We used same envelopes for all 400 simulations. These envelopes were made from 99 simulation of the model in the tested hypothesis. Thus we did not perform the parameters estimation part. This simplification caused acceleration of the procedure and it caused that the numbers of rejections can be a bit higher than when it would be done without such simplification.

The numbers of rejections of the Boolean model hypothesis are summarized in Table 2.

Table 2

5. Real data study - heather incidence

Figure 2

The heather incidence data (see Figure 2) was studied in Diggle (1981). Since individual heather plants grow into hemispherical bushes, reaching a maximum radius of about 50 cm, the Boolean model of disks was chosen in Diggle (1981) as a model for the real data description. For the distribution of the disks radii was fitted the shifted Weibull distribution with parameters (0.0281, 0.8471, 144.7). The intensity λ was estimated to 221 disks per unit area. The realization of the Boolean model fitted in Diggle (1981) can be seen on Figure 3. As we can see from the realization and as was mentioned in Diggle (1981) and Cressie (1997), p. 765 the Boolean model is unsatisfactory for the data description because the data contains fewer patches than the Boolean model. But there was performed simulation based test (described here in Section 2.3) which did not reject the Boolean model hypothesis in Diggle (1981).

Figure 3

We chose this data set to show how the proposed test works for the real data. Since the data set has the resolution 200×100, we used, for the dilation Ξ_ε, the discretized disks with radii $\varepsilon_1, \ldots \varepsilon_{24}$ evenly spread between 1 and 6 pixels of the image. (Greater disks produce after dilation just a white rectangle.) First we performed the proposed test on the Boolean model hypothesis where the Boolean model is fitted as above. As we can see from Figure 4, the Boolean model hypothesis is clearly rejected by this test.

Figure 4

Thus, we tried to fit another model for the heather incidence data. We chose the Matérn cluster model (for details see Stoyan et al. (1995)) for the disks centers. For the distribution of the radii we chose the same Weibull distribution as above. Thus only 3 parameters remains to estimate. The intensity λ, the maximum radius of the clusters R and the mean number of disks centers in the cluster μ. Since this work is not concerned on the estimation of the model parameters we estimated these three parameters in very naive way. The parameters R and μ was guessed from the data and λ was set that the area fractions of the data and the simulations correspond to each other. The resulted naive estimates are $R = 0.03$, $\mu = 1$ and $\lambda = 350$. The realization of this model, shown on Figure 3, reflects a right number of patches. And the proposed test, shown on Figure 5, does not reject the Matérn cluster model assumption, because only 3 observations fall outside of the envelopes.

Figure 5

6. Discussion

The comparison of the tests shows that the simulation-based tests are valuable procedure even for testing the Boolean model hypothesis where other specific tests are available. The proposed test seems to be a sensitive tool for distinguishing the interaction between particles of the binary image. On the other side, the proposed test is not sensitive to the wrong assumption of the distribution of the primary grain. The computer programme prepared for public use is published on www.pf.jcu.cz/~mrkvicka/math.

7. Acknowledgements

The author is grateful to Jakub Krafka for the implementation of the algorithm in C++.

References

[1] N. A. C. Cressie: *Statistics for Spatial Data. Revised Edition.* J. Wiley, New York 1997

[2] T. Dam, J. B. Sorensen, M. H. Thomsen: *Spatial Point Processes, Models, Simulation and Statistical Inference.* Master thesis,Alborg 1999

[3] P. J. Diggle: Binary Mosaics and the Spatial Pattern of Heather. *Biometrics* **37** (1981), 531–539

[4] I. Molchanov: *Statistics of the Boolean Model for Practitionars and Mathematicians.* J. Wiley, New York 1997

[5] J. Moller, R. P. Waagepetersen: *Statistical Inference and Simulation for Spatial Point Processes.* Chapman & Hall/CRC, London 2004

[6] T. Mrkvička, J. Rataj: On estimation of intrinsic volume densities of stationary random closed sets. *Stochastic Process. Appl.* submitted
[7] W. Nagel, J. Ohser, K. Pischang: An integral-geometric approach for the Euler-Poincaré characteristic of spatial images. *J. Microsc.* **198** (2000), 54–62
[8] Ohser J., Mücklich F. 2000 *Statistical Analysis of Microstructures in Materials Science*, John Wiley & Sons, Chichester.
[9] J. Rataj: On estimation of the Euler number by projections of thin slabs. *Adv. Appl. Prob* **36** (2004), 715–724
[10] J. Rataj: Estimation of intrinsic volumes from parallel neighbourhoods. *Rend. Circ. Mat. Palermo* (submitted)
[11] V. Schmidt, E. Spodarev: Joint estimators for the specific intrinsic volumes of stationary random sets. *Stochastic Process. Appl.* **115** (2005), 959–981
[12] R. Schneider: *Convex Bodies. The Brunn-Minkowski Theory.* Cambridge Univ. Press, Cambridge 1993
[13] R. Schneider, W. Weil: *Stochastic Geometry.* Teubner, Stuttgart 2000.
[14] D. Stoyan, W.S. Kendall, J. Mecke: Stochastic Geometry and Its Applications, Second Edition. J. Wiley, Chichester 1995

List of Figures

TABLE 1. The numbers of successive rejections of the Boolean model hypothesis from 30 simulations.

	Cluster model	Regular model
1) Laslett's test	2	0
2) Linear and quadratic CDF	11	19
3) Simulation-based test - quadratic CDF	22	22
4) Proposed test	27	19

TABLE 2. The numbers of rejections of the Boolean model hypothesis from 100 simulations under a different wrong assumptions.

Wrong assumption	rejections
0) no wrong assumption	7
1) shape of the particles	8
2) distribution of the particles sizes	7
3) both	10

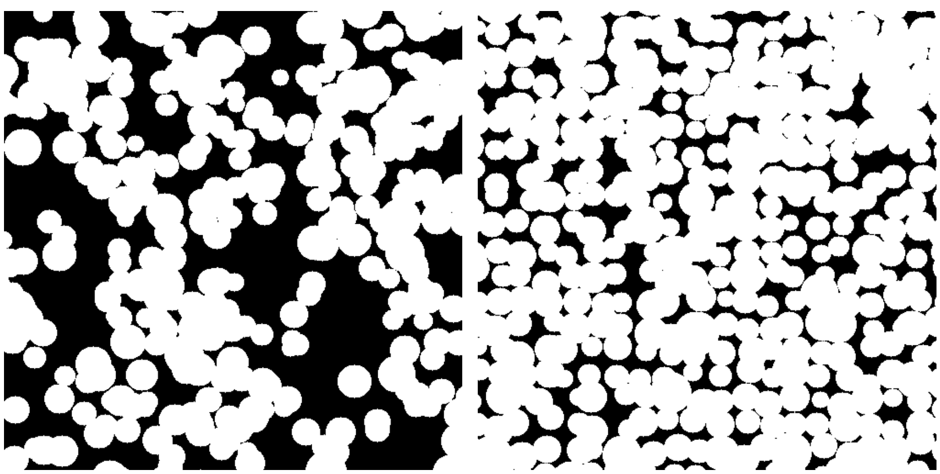

FIGURE 1. Left: One observation of the chosen cluster model in the observation window $W = 500 \times 500$ pixels. Right: One observation of the chosen regular model in the observation window $W = 500 \times 500$ pixels.

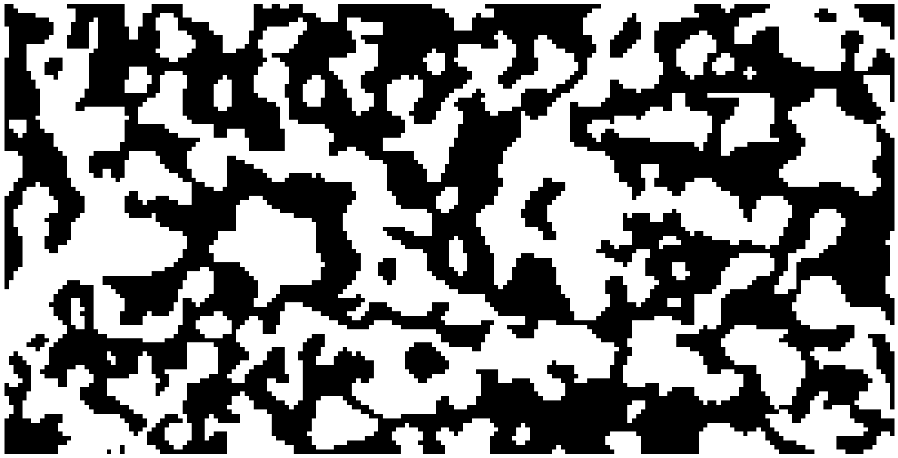

FIGURE 2. The heather incidence data in a rectangular area 2×1 collected with the resolution 200×100 binary pixels. Heather is the white area.

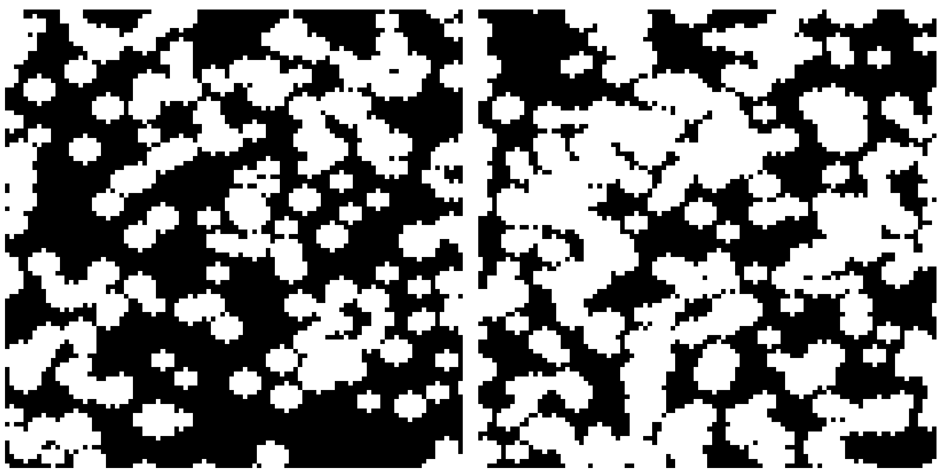

FIGURE 3. Left: One realization of Boolean model of the disks with parameters fitted in [3] in the resolution 100×100 pixels. Right: One realization of fitted Matérn Cluster model of the disks in the resolution 100×100 pixels.

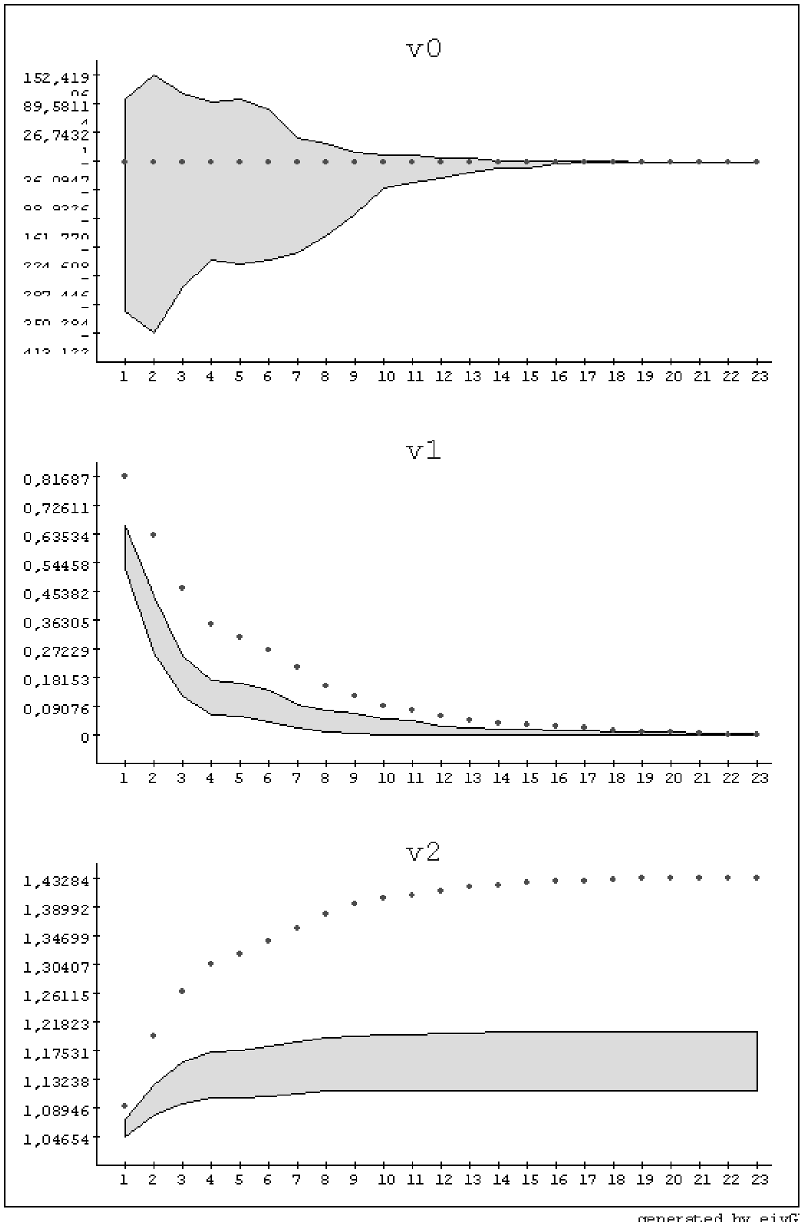

FIGURE 4. The results of the proposed test on the Boolean model assumption.

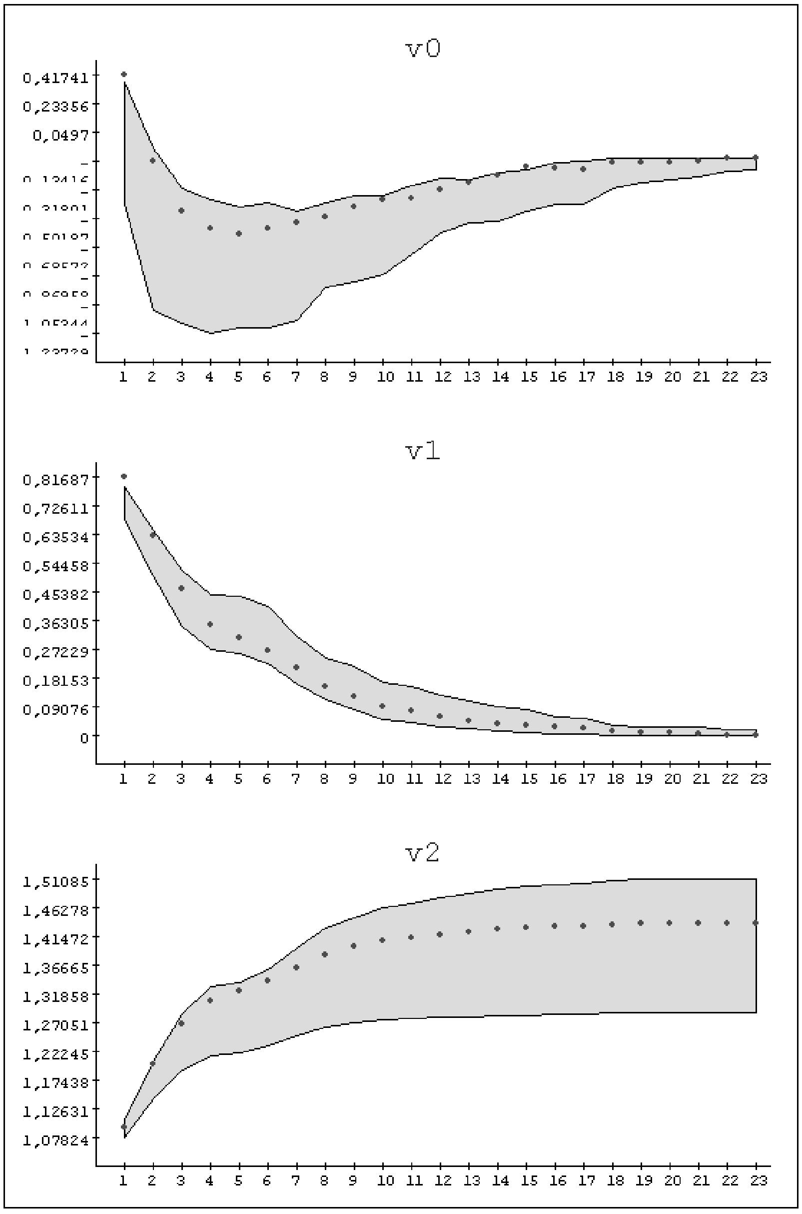

FIGURE 5. The results of the proposed test on the Matérn cluster model assumption.