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RESEARCH REPORT

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CONVEXITY and DEPENDENCE in CHANCE-CONSTRAINED PROGRAMMING

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Abstract. The question of convexity of feasible set with probabilistic constraints was investigated recently in the literature (Henrion, Strugarek, 2006). Under the assumption of independence of the random variables, the authors show that the convexity property of chance constraints holds if the problem possesses a suitable combination of a concavity property of the constraint mappings, and a sufficient decrease of the marginal densities. Adopting settings of the paper, we leave out the assumption of independence in favour of the particular weak dependence property, and show that under such assumption, the original nonconvex problem can be approximated by a convex one, widely desirable for algorithmic and theoretical purposes.

Keywords: Stochastic programming, chance-constrained programming, stability, convexity, weak dependence.

AMS classification: 90 C 15

1 Introduction

Consider a chance-constrained problem of the form

$$\min F_0(x) \text{ subject to } \Pr\{h(x;\xi) \geq 0\} \geq p \tag{1}$$

where $x \in \mathbb{R}^n$ is a decision vector, $\xi : \Omega \to \mathbb{R}^s$ is an s-dimensional random vector defined on some probability space $(\Omega, \mathcal{A}, \Pr)$, $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^d$ is a vector-valued mapping, and $p \in [0; 1]$ is a (prescribed) probability level. Denote $\mu = \Pr \circ \xi^{-1}$ the distribution of the random vector ξ, $F = F_\mu$ its distribution function; further for each $x \in \mathbb{R}^n$ define $H(x) = \{\xi \in \mathbb{R}^s : h(x;\xi) \geq 0\}$ a set of realizations satisfying the (original) constraints. The set of feasible solutions of (1) can be then written as

$$M(p) := \left\{x \in \mathbb{R}^n | \Pr\{\xi \in H(x)\} = \mu(H(x)) \geq p\right\}$$

and the whole problem simply as

$$\min F_0(x) \text{ subject to } x \in M(p)$$

Such kind of problems are common in engineering and economic practice and applications; well known are the applications in energy, water resources, production and inventory, telecommunications, and others. See [5], [9], and references therein for details and further applications of chance-constrained programming.

A matter of importance both in theoretical and practical applications of chance-constrained programming is to determine when the set $M(p)$ of feasible solutions is convex. It is trivially known that the sets $M(0)$, $M(1)$ (i. e., the set without probabilistic constraints and the set of constraints satisfied almost surely) are convex if $h(\cdot, \xi)$ are convex functions for all $\xi \in \mathbb{R}^s$. Classical result of Prékopa [2] (see also [3], [8], [4], [5] and references therein) states the following:

Proposition 1 ([5], Theorems 2.5 and 2.11)**.** *If μ is absolutely continuous (with respect to Lebesgue measure), log-concave (or r-concave for $r \geq -1/s$) measure, and the one-dimensional components of h are quasi-concave functions of (x, ξ) then $M(p)$ is convex set.*

A log-concave (or r-concave) measure is implied by a log-concave (or $\frac{r}{1-rs}$-concave) density. It is known (see the above references again) that many prominent multivariate distributions satisfy the condition, hence many chance-constrained problems can be solved by means of convex optimization.

The quasi-concavity property is not preserved under addition. For example, consider a problem with random right-hand side in the form

$$\min F_0(x) \text{ subject to } \Pr\{g(x) \geq \xi\} \geq p. \tag{2}$$

Problem (2) falls into the frame of (1) if we set $h(x; \xi) = g(x) - \xi$. In order to have $h(x)$ quasi-concave, it is not sufficient to have $g(x)$ quasi-concave and we require $g(x)$ to be concave. Recently, Henrion and Strugarek [1] proposed an alternative approach to deal with this problem: their idea is to relax concavity condition of g and make more stringent concavity condition on the probability distribution μ. We recall this in Section 2. In Section 3 we relax the condition of independence that authors of [1] require and show that, under modified assumptions, their results still remain valid. Section 4 introduces stability properties of the optimal values of the "dependent" and "independent" problems.

2 Concavity and density decrease

2.1 r-concave functions

We recall the definition of the r-concave function (see [5], Definition 2.3):

Definition 2. A function $g : \mathbb{R}^m \to (0; +\infty)$ is called r-concave for some $r \in [-\infty; +\infty]$ if

$$g(\lambda x + (1-\lambda)y) \geq \left[\lambda g^r(x) + (1-\lambda) g^r(y)\right]^{1/r}$$

for all $x, y \in \mathbb{R}^m$ and all $\lambda \in [0; 1]$. Cases $r = -\infty, 0, +\infty$ have to be treated by continuity.

We sumarize some interesting cases in Table 2.1. For our purpose the most important are functions that are r-concave for $r \leq 1$. Note that if g is r^*-concave, it is also r-concave for all $r \leq r^*$.

$r = +\infty$	$\ldots\ g(\lambda x + (1-\lambda)y) \geq \max\{g(x), g(y)\}$
$r \in (1; +\infty)$	$\ldots\ g^r$ is concave
$r = 1$	$\ldots\ g$ is concave
$r = 0$	$\ldots\ g$ is **log-concave** ($\log f$ is concave): $g(\lambda x + (1-\lambda)y) \geq g^{\lambda}(x) g^{1-\lambda}(y)$
$r < 0$	$\ldots\ g^r$ is convex
$r = -\infty$	$\ldots\ g$ is **quasi-concave:** $g(\lambda x + (1-\lambda)y) \geq \min\{g(x), g(y)\}$

Table 1: r-concave functions.

2.2 r-decreasing densities

Here we adopt the definition of [1] for a so-called r-decreasing function.

Definition 3 ([1], Definition 2.2). A function $f : \mathbb{R} \to \mathbb{R}$ is called r-decreasing for some $r \in \mathbb{R}$ if

(i) it is continuous on $(0; +\infty)$, and

(ii) there exists a threshold $t* > 0$ such that $t^r f(t)$ is strictly decreasing for all $t > t*$.

If $r = 0$ the function f is strictly decreasing in the classical sense. As for r-concave functions, if f is r^*-decreasing then it is r-decreasing for all $r \leq r^*$. In [1] it is shown that if marginal densities of the distribution μ are $(r+1)$-decreasing (for some $r > 0$), then the mapping $t \mapsto F(t^{-1/r})$ (F beeing the corresponding distribution function) is concave for $t \in (0; t^{*-r})$, which is further shown to be sufficient to ensure convexity property of the problem (2):

Theorem 4 ([1], Theorem 3.1). *If*

(i) there exist $r_i > 0$ such that the components g_i of g are $(-r_i)$-concave,

(ii) the components ξ_i of ξ have $r_i + 1$-decreasing densities, and

(iii) the components ξ_i of ξ are independently distributed,

then $M(p)$ is convex for all $p > p^* := \max_i F_i(t_i^*)$ where F_i denotes the distribution function of ξ_i and t_i^* refer to the definition of $r_i + 1$-decreasing probability density.

As shown in [1], required limit constants p^* for prominent one-dimensional distributions are not really high and so Theorem 4 can be directly used in further applications.

3 Weak dependence of the rows

In the sequel we ask for the relaxation of the independence condition (iii) in Theorem 4. To do this, we define an α' coefficient of the dependence by the following definition.

Definition 5. For a random vector ξ we define α' coefficient of dependence as

$$\alpha' := \sup_z \big| F(z) - \prod_i F_i(z_i) \big| \tag{3}$$

where F is the distribution function of the vector ξ, F_i are the corresponding one-dimensional marginal distribution functions and $z = (z_1, \ldots, z_s) \in \mathbb{R}^s$.

If $\alpha' = 0$, ξ_i's are independent random variables. Our α'-coefficient is actually a modified version of the known strong-mixing coefficient (see e. g. [10]). The classical strong-mixing (α-mixing) coefficient is generally defined for two σ-fields $\mathcal{B}_1$, $\mathcal{B}_2 \subset \mathcal{A}$ by

$$\alpha(\mathcal{B}_1, \mathcal{B}_2) := \sup_{A \in \mathcal{B}_1, B \in \mathcal{B}_2} |\Pr(A \cap B) - \Pr(A)\Pr(B)|$$

If $\mathcal{B}_1, \mathcal{B}_2$ are σ-fields generated by two random vectors ξ_1, ξ_2, we say that the two vectors are α-dependent. For our purposes we extend the definition to more than two random variables (this is natural) and use their distribution functions instead of generated σ-fields. Of course, if some variables ξ_1, ξ_2 are α-dependent (in the strong-mixing sense), they are also α'-dependent in the sense of Definition 5.

Convention. To simplify the notation, we drop the prime symbol $'$ from Definition 5 and use the notion of α-dependence in the sense of Definition 5 for the remaining part of the paper.

In what follows we allow for a small structural dependence introduced by a small value of α. Recall that the set of feasible solutions of (2) can be written as

$$M(p) := \big\{ x \in X | F(g(x)) \geq p \big\}$$

where F is the distribution function of the random right-hand side ξ. Further, denote

$$M'(p) = \big\{ x \in X | \prod_{i=1}^{s} F_i(g_i(x)) \geq p \big\}$$

where F_i are one-dimensional marginal distribution functions of F.

If the components ξ_i of ξ are independently distributed, the two sets are equal:

$$M(p) = M'(p)$$

This is not true in case of weak dependence, but the following proposition is valid:

Proposition 6. *If the components ξ_i of ξ in* (2) *are α-dependent (in the sense of Definition 5) then*

$$M'(p+\alpha) \subset M(p) \subset M'(p-\alpha) \subset M(p-2\alpha). \tag{4}$$

Proof. If the components of ξ are α-dependent, we have

$$|F(g(x)) - \prod_i F_i(g_i(x))| \leq \alpha, \text{ i.e.}$$

$$\prod_i F_i(g_i(x)) - \alpha \leq F(g(x)) \leq \prod_i F_i(g_i(x)) + \alpha$$

The second inequation together with the definition of $M(p)$ implies

$$p \leq F(g(x)) \leq \prod_i F_i(g_i(x)) + \alpha$$

$$\prod_i F_i(g_i(x)) \geq p - \alpha,$$

that is

$$M(p) \subset M'(p-\alpha) \tag{5}$$

Similarly the first inequation of (3) yields

$$p - \alpha \leq \prod_i F_i(g_i(x)) \leq F(g(x)) + \alpha$$

$$F(g(x)) \geq p - 2\alpha$$

hence

$$M'(p-\alpha) \subset M(p-2\alpha). \tag{6}$$

Combining equations (5) and (6) we obtain the whole chain of inequalities

$$M'(p+\alpha) \subset M(p) \subset M'(p-\alpha) \subset M(p-2\alpha) \tag{7}$$

□

If p is sufficiently high, then possibly non-convex $M(p)$ is bounded from both side by convex sets:

Theorem 7. *If*

(i) there exist $r_i > 0$ such that the components g_i of g are $(-r_i)$-concave,

(ii) the components ξ_i of ξ have $r_i + 1$-decreasing densities,

(iii) the components ξ_i of ξ are α-dependently distributed, and

(iv) $p > \max_i F_i(t_i^*) + \alpha$

then $M(p)$ is bounded (from both sides) by ***convex*** *sets $M'(p+\alpha)$ and $M'(p-\alpha)$.*

Proof. The last will be proved modifying the proof of Theorem 4. Due to the assumptions, for $x, y \in M'(p-\alpha)$ we have

$$0 \leq F_i(t^*) < p - \alpha \leq F_i(g_i(x)) < 1$$

and the same inequalities are valid for x replaced by y. The second inequality follows simply from the assumption (iv), the last one is proved by Lemma 3.1 of [1]. By the continuity of marginal distributions functions and quantile properties we obtain

$$0 < t_i^* < F_i^{-1}(F_i(g_i(x))) \leq g_i(x)$$

and the same with y. The remaining part of the proof is the same as in [1], in particular we have for $\lambda \in [0; 1]$

$$\begin{aligned}
&\prod_{i=1}^{m} F_i(g_i(\lambda x + (1-\lambda)y)) \\
&\quad \geq \prod_{i=1}^{m} [F_i(F_i^{-1}(F_i(g_i(x))))]^{\lambda} [F_i(F_i^{-1}(F_i(g_i(y))))]^{1-\lambda} \\
&\quad = [\prod_{i=1}^{m} F_i(g_i(x))]^{\lambda} [F_i(g_i(y))]^{1-\lambda} \\
&\quad \geq (p-\alpha)^{\lambda} (p-\alpha)^{1-\lambda} \\
&\quad = p - \alpha
\end{aligned}$$

hence $\lambda x + (1-\lambda)y \in M'(p-\alpha)$. The other bound follows directly from Proposition 6. □

Theorem 7 has great impact also on numerical solutions of "dependent problems." If we (somehow) know that the components of ξ are α-dependent so that α is small enough (at least such that condition (iv) of 7 is fulfilled), then, solve two convex, "independent" problems

$$\min F_0(x) \text{ subject to } x \in M'(p-\alpha), \text{ and} \tag{8}$$

$$\min F_0(x) \text{ subject to } x \in M'(p+\alpha). \tag{9}$$

The solutions of these problems (or a solution of an "independent" problem with $M'(p)$) are good approximations of the original problem in the meaning that both optimal values of the problems are lower and upper bounds of the optimal value of the original problem. An insight into stability properties of the problem is provided by the following section.

4 Stability properties

We have already stated that optimal value of the problem (2) is bounded by optimal values of the problems (8) and (9). Under additional assumptions on objective and constraint functions, we can prove Theorem 9. Before do that we recall the definition of (special) metric regularity.

Definition 8. Set valued mapping $x \mapsto \{y \in \mathbb{R} | x \in M'(p-y)\}$ is said to be *metrically regular* at some pair $(\bar{x}, 0) \in \mathbb{R}^n \times \mathbb{R}$ if there exist $\varepsilon > 0$ and $a > 0$ such that

$$\mathrm{dist}(x, M'(p-y)) \leq a \max\{0, p + y - \prod_{i=1}^{s} F_i(g_i(x))\}$$

for all $x \in \mathbb{B}_\varepsilon(\bar{x}), y \in \mathbb{B}_\varepsilon(0)$ where $B_\varepsilon(x)$ is ε-neighbourhood of x.

For the general definition of metric regularity condition and its properties see [6].

Theorem 9. *Consider the problem* (2) *and let*

(i) assumption (i)–(iv) of Theorem 7 be fulfilled;

(ii) F_0 be Lipschitz continuous function on $\mathbb{R}^n$;

(iii) the mapping $x \mapsto \{y \in \mathbb{R} | x \in M'(p+\alpha-y)\}$ be metrically regular at $(\bar{x}'(p+\alpha), 0)$;

(iv) α-dependence coefficient satisfie $\alpha < \frac{1}{2}\varepsilon$, where ε is provided by metric regularity condition (iii).

Then there exists a constant $L > 0$ such that

$$|\varphi'(p+\alpha) - \varphi(p)| \leq L \max\{0, p+\alpha - \prod F_i(g_i(\bar{x}'(p-\alpha)))\} \tag{10}$$

is fulfilled. Here, $\varphi(p)$, $\varphi'(p+\alpha)$ are the optimal values of (2) and (9) respectively, and $\bar{x}'(p-\alpha)$, $\bar{x}'(p+\alpha)$ are the optimal solutions of (8) and (9) respectively.

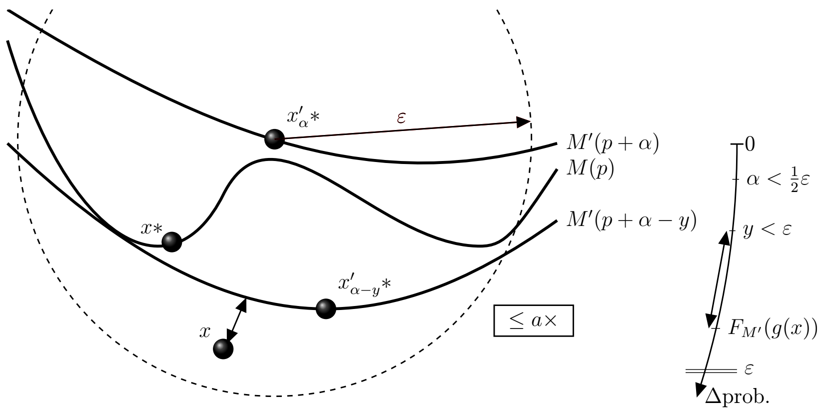

Figure 1: Metric regularity and bounded feasibility sets.

Proof. Under the assumptions of the theorem

$$\begin{aligned} &|\varphi'(p+\alpha) - \varphi(p)| \\ &\quad \leq |\varphi'(p+\alpha) - \varphi'(p-\alpha)| \\ &\quad \leq L\|\bar{x}'(p+\alpha) - \bar{x}'(p-\alpha)\| \end{aligned}$$

If $\alpha < \frac{1}{2}\varepsilon$, the metric regularity condition imply

$$\begin{aligned} &\|\bar{x}'(p+\alpha) - \bar{x}'(p-\alpha)\| \\ &\quad \leq a \max\{0, p+\alpha - \prod F_i(g_i(\bar{x}'(p-\alpha))\} \end{aligned}$$

Combining the two inequalities yields the assertion. □

The concept of Theorem 9 is illustrated in Figure 1. The stability of optimal solutions is more complicated thing that require some kind of growth condition (see [7]) and is subject of further research.

5 Conclusion

Convexity of feasible sets in chance-constrained programs is considered as crucial for both theoretical and algorithmic purposes. We have shown that weak dependence of the rows presents no actual difficulty; under traditional condition on Lipschitz continuity of objective function, the optimal value of the problem remain stable even for weak (α-)dependent case. This has a significant impact on practical solutions: we can solve the independent problem even if we are not sure that the true independence is actually present in the problem. The price of such solution will not notably differ if the dependence level is small.

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