Stability and metrics in stochastic programming problems

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Abstract. Some of the basic results concerning qualitative and quantitative stability of optimal values and optimal solution sets to stochastic programming problems are presented: in detail, when the underlying probability distribution varies in some metric space of probability measures. We have made a list of commonly used probability metrics and some examples of their use. The general theorems are applied to the recourse problems especially.

Introduction

Building a stochastic optimization model, there is in most cases assumed that the underlaying probability distribution is fully known. Usually we do not have this information in practice – we are dealing with some kind of approximation of the true distribution, e. g. an empirical distribution. In other cases, we may know the true distribution, but a complexity of the problem (e. g. a necessity of the numerical computation of multidimensional integrals) disallows us to obtain any solution; in such cases we usually replace the true distribution with some simpler, generally discrete one (and speak about *scenarios*).

In all the mentioned examples, a question arises how good is the used approximation. It happens that even a small change to the original probability measure may lead to large errors in optimal values or optimal solutions. This motivates an introduction to the *stability analysis* of stochastic programming problems *with respect to changes in probability measure*. Of course, it is necessary to quantify this changes by a suitable distance, i. e., to provide suitable *metrics* to the considered probability space.

This article presents and summarizes some older and recent results to the just posed problem, together with a survey of probability metrics commonly used in touch with stochastic optimization problems.

Problem formulation

Consider a general decision model of the form

$$\inf_{x\in\mathbb{R}^n} \int_{\Xi} f_0(x;\xi)\ \mu(\mathrm{d}\xi) \quad \text{subject to} \quad \mu\big\{\xi\in\Xi : x\in X(\xi)\big\} \geq p_0 \tag{1}$$

where $\mu \in \mathcal{P}(\Xi)$, the space of all Borel probability measures defined on some Borel set $\Xi \subset \mathbb{R}^s$, $f_0\colon \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $X\colon \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ (both measurables), and $p_0 \in [0;1]$ is a prescribed probability level. Stochastic programs with recourse, as well as with probabilistic constraints fit into (1). Denote $\varphi(\mu)$ the optimal value and $\psi(\mu)$ the optimal solution set of the problem (1). Let now replace the original distribution μ with another one, denoted $\nu \in \mathcal{P}(\Xi)$ (e. g., empirical one, discretized version, etc.). Our intention is to find how the optimal value and optimal solution set have changed. In particular, we want to set up conditions under which $\varphi(\cdot)$ and $\psi(\cdot)$ are (semi-) continuous at μ, and try to quantify this continuity, e. g. by some Lipschitz property of the form $|\varphi(\mu) - \varphi(\nu)| \leq L \cdot d(\mu,\nu)$, where d is a suitably chosen probability metrics.

Common probability metrics

In this section, we introduce a list of common metrics, defined on a given space of probability measures (a subset of $\mathcal{P}(\Xi)$). Not all metrics are usable in all cases that could happen across the stochastic optimization — we will cite several examples, where the application of presented metrics fails.

Kolmogorov metrics

The *Kolmogorov (Kolmogorov–Smirnov) metrics* is defined by

$$\mathcal{K}(\mu,\nu) := \sup_{z\in\Xi}|F(z) - G(z)|,$$

F, G are distribution functions corresponding respectively to $\mu, \nu \in \mathcal{P}(\Xi)$. A considerable advantage of this metrics is a computational simplicity when it is applied to workaday optimization problems. On the other hand, the Kolmogorov metrics can give us unusable results: e. g., when approximating discrete distribution with unknown mass points, as seen on the figure 1 for the special case of degenerated distribution. Another example of a mismatched application of the Kolmogorov metrics can be found in Dupačová, Römisch [1998]: we want to measure changes in optimal value, originated by including an additional scenario to the current problem. Kolmogorov metrics does not reflect (under cited conditions) the chosen position of such scenario, as one would expect from a suitable metrics.

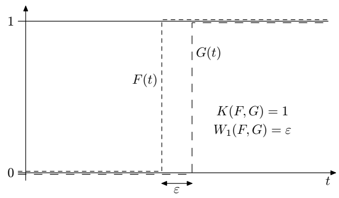

Figure 1. Approximation of the discrete distribution with unknown mass points

Wasserstein metrics

Let

$$\mathcal{P}_p(\Xi) := \left\{\nu \in \mathcal{P}(\Xi) : \int_\Xi \|\xi\|^p \nu(\mathrm{d}\xi) < +\infty\right\}$$

for $p \in [1; +\infty)$. The *p-Wasserstein metrics* (also called the Kantorovich metrics) is defined for $\mu, \nu \in \mathcal{P}(\Xi)$ by

$$W_p(\mu,\nu) := \left[\inf\left\{\int_{\Xi\times\Xi} \|\xi - \tilde{\xi}\|^p \; \eta(\mathrm{d}\xi \times \mathrm{d}\tilde{\xi}) : \eta \in D(\mu,\nu)\right\}\right]^{1/p},$$

where $D(\mu,\nu)$ is the set of all probability measures (of $\mathcal{P}(\Xi\times\Xi)$), for which μ and ν are marginal distributions. In one-dimensional case, the situation is less complicated: it can be shown (see Vallander [1973]), that

$$W_p(\mu,\nu) = \left(\int_0^1 |F^{-1}(t) - G^{-1}(t)|^p \mathrm{d}t\right)^{1/p} \quad \text{and} \quad W_1(\mu,\nu) = \int_{-\infty}^{+\infty} |F(t) - G(t)|\mathrm{d}t$$

It is evident that the main disadvantage of this metrics is its computational difficulty, apart from a majority of situations when we cannot get its value at all. Even being in one-dimensional space, we have to take care of cases, where the integral of $|F(z) - G(z)|$ could rise above arbitrary or rather high value, as cited in Houda [2001] (e. g., if $F(z) := \varepsilon/(1-z)$ on $(-\infty; 0)$ for $\varepsilon \in (0; 1)$, $G(z) := 0$ on the same interval, $F(z), G(z)$ arbitrary on $[0; +\infty)$, the Wasserstein metrics takes a value of $+\infty$). At the end of this part, note that $(P_p(\Xi), W_p)$ is metric space, and W_p metrizes the weak convergence on $\mathcal{P}_p(\Xi)$). We refer to Römisch, Schultz [1991b] for details.

Zolotarev's pseudometrics and Fortet-Mourier metrics

The next is introduced in ZOLOTAREV [1981] and RACHEV [1991]. Let $\mathcal{F}$ be a class of measurables functions $f : \Xi \to \overline{\mathbb{R}}$. Consider a class $\mathcal{P}_{\mathcal{F}} \subset \mathcal{P}(\Xi)$ and $\mu, \nu \in \mathcal{P}_{\mathcal{F}}$. The *Zolotarev's pseudometrics* (or *distance having ζ-structure*) on $\mathcal{P}_{\mathcal{F}}$ is defined by

$$d_{\mathcal{F}}(\mu,\nu) := \sup_{f\in\mathcal{F}} \left| \int_\Xi f(\xi)\mu(\mathrm{d}\xi) - \int_\Xi f(\xi)\nu(\mathrm{d}\xi) \right|$$

It is a metrics if $d_{\mathcal{F}}$ is finite and the class $\mathcal{F}$ is rich enough to preserve that $d_{\mathcal{F}}(\mu,\nu) = 0$ implies $\mu = \nu$. As an example, the first order Wasserstein metrics follows this frame, thus it is a distance having ζ-structure: the appropriate class is $\mathcal{F}_1 := \{f : \Xi \to \mathbb{R} : L_1(f) \leq 1\}$, where the first order Lipschitz constant is defined by $L_1(f) := \inf\{L : |f(\xi) - f(\tilde{\xi})| \leq L\|\xi - \tilde{\xi}\| \forall \xi, \tilde{\xi} \in \Xi\}$.

In many cases, we deal with functions that grow faster than linearly. We therefore define a Lipschitz constant of order $p \in [1; +\infty)$ by
$L_p(f) := \inf\{L : |f(\xi) - f(\tilde{\xi})| \leq L\|\xi - \tilde{\xi}\| \max\{1, \|\xi\|^{p-1}, \|\tilde{\xi}\|^{p-1}\}\ \forall \xi, \tilde{\xi} \in \Xi\}$ and the corresponding class $\mathcal{F}_p := \{f : \Xi \to \mathbb{R} : L_p(f) \leq 1\}$. The *Fortet–Mourier metrics* is then defined by $\zeta_p(\mu,\nu) := d_{\mathcal{F}_p}(\mu,\nu)$. Apparently, it coincides with the Wasserstein metrics for the case of $p = 1$.

Bounded Lipschitz β-metrics

The *bounded Lipschitz β-metrics* has been used in the first papers concerning our theme (see the third section of the paper). It is a distance having ζ-structure with class $\mathcal{F}_{\mathrm{BL}} := \{f : \Xi \to \mathbb{R} : \|f\|_{\mathrm{BL}} \leq 1\}$, where $\|f\|_{\mathrm{BL}} := \sup_{\xi\in\Xi}\|f(\xi)\| + \sup_{\substack{\xi,\tilde{\xi}\in\Xi \\ \xi\neq\tilde{\xi}}} \frac{|f(\xi)-f(\tilde{\xi})|}{\|\xi-\tilde{\xi}\|}$. It also metrizes the topology of weak convergence on $\mathcal{P}(\Xi)$.

α-metrics

The *α-metrics* applies when the stability of problems with probability constraints is considered. It is defined for $\mu, \nu \in \mathcal{P}(\Xi)$ by $\alpha_{\mathcal{B}}(\mu,\nu) := \sup_{B\in\mathcal{B}}|\mu(B) - \nu(B)|$, where $\mathcal{B}$ is a subset of the Borel σ-field $\mathbb{B}(\Xi)$. If $\mathcal{B} = \{(-\infty; z]; z \in \mathbb{R}^s\}$ then this metrics coincides with the Kolmogorov metrics. For details see RÖMISCH, SCHULTZ [1991a].

Stability results

Stability for recourse problems

The β-metrics applied to a study of stability has appeared in a paper of RÖMISCH, WAKOLBINGER [1987] and has been extended later in RÖMISCH, SCHULTZ [1991a]. Theirs main results follow from the next general theorem proved in the first of the mentioned papers.

Theorem 1 *For $p \in [1; +\infty)$ and $\nu \in \mathcal{P}(\Xi)$, let define a generalized moment $M_p(\nu) := \left(\int_\Xi \big(L(\|\xi\|)\cdot\|\xi\|\big)^p \nu(\mathrm{d}\xi)\right)^{1/p}$, where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is some continuous increasing function. If any function $f : \Xi \to \mathbb{R}$ satisfies $|f(\xi) - f(\tilde{\xi})| \leq L\big(\max\{\|\xi\|, \|\tilde{\xi}\|\}\big)\|\xi - \tilde{\xi}\|$ then there exists $C > 0$ such that for all $p \in (1; +\infty)$ and $\mu, \nu \in \mathcal{P}(\Xi)$ we have*

$$\left| \int_\Xi f(\xi)\mu(\mathrm{d}\xi) - \int_\Xi f(\xi)\nu(\mathrm{d}\xi) \right| \leq C(1 + M_p(\mu) + M_p(\nu)) \cdot \beta(\mu,\nu)^{1-\frac{1}{p}}.y$$

The exponent $1 - 1/p$ in the above Hölder property is shown as optimal. The authors give also some applications, among them a problem with the complete fixed linear recourse: minimize

$$F_1(x;\mu) := c^T x + \int_\Xi h_1(x;\xi)\mu(\mathrm{d}\xi) \quad \text{subject to} \quad x \in X_0, \tag{2}$$

where $X_0 \subset \mathbb{R}^n$ is nonempty convex polyhedron, $c \in \mathbb{R}^n$, $h_1(x;\xi) := \min_{y\in\mathbb{R}^m}\{q^T y : Wy = a - Tx, y \geq 0\}$ (linear recourse), and $\xi = (q, a, T)$, with assumptions (A1) $\{y \in \mathbb{R}^m : Wy = b, b \geq 0\} \neq \emptyset$ for all $b \in \mathbb{R}^r$, and (A2) $\{u \in \mathbb{R}^r : W^T u \leq q\} \neq \emptyset$. Denoting $\mathcal{P}_{p,K}(\Xi) := \big\{\nu \in \mathcal{P}(\Xi) : \int_\Xi \|\xi\|^p \mu(\mathrm{d}\xi) \leq K\big\}$ for $p \in [1; +\infty)$ and $K > 0$, the theorem reads:

Theorem 2 *Let $p > 1$, $K > 0$ and fix $\mu \in \mathcal{P}_{2p,K}(\Xi)$. Assume further that $\psi(\mu)$ is nonempty and bounded. Then ψ is (Berge) u. s. c. at μ w. r. t. $(\mathcal{P}_{2p,K}(\Xi), \beta)$, and there exist constants $\delta_1 > 0$, $L_1 > 0$ such that for any $\nu \in \mathcal{P}_{2p,K}(\Xi)$ fulfilling $\beta(\mu,\nu) < \delta_1$ we have that $\psi(\nu) \neq \emptyset$ and*

$$|\varphi(\mu) - \varphi(\nu)| \leq L_1 \beta(\mu,\nu)^{1-\frac{1}{p}}.$$

Note, in the previous theorem, the moment condition of order $2p$. In their paper, Römisch, Schultz [1991b] have weakened significantly this assumption, and got better (Lipschitz) estimates with the Wasserstein metrics. The next theorem is taken from Römisch, Schultz [1993], where the special case of complete fixed linear recourse is explored.

Theorem 3 *Consider the problem* (2) *with (A1), (A2) and assume (A3): q or (a, T) are non-stochastic. Let $\mu \in \mathcal{P}_1(\Xi)$ and assume that $\psi(\mu)$ is nonempty and bounded set. Then ψ is (Berge) u. s. c. at μ w. r. t. $(\mathcal{P}_1(\Xi), W_1)$ and there exist constants $\delta_1 > 0$, $L_1 > 0$ such that for any $\nu \in \mathcal{P}_1(\Xi)$ for which $W_1(\mu,\nu) < \delta_1$ we have that $\psi(\nu) \neq \emptyset$ and*

$$|\varphi(\mu) - \varphi(\nu)| \leq L_1\, W_1(\mu,\nu)$$

If we do not take into account the assumption (A3), we would get a similar proposition with $\mathcal{P}_2(\Xi)$ and the metrics W_2. By modifying assumptions of the theorem 3 and its proof, we can come at a possible nonlinear generalization, as in Houda [2001]:

Theorem 4 *Consider a program of minimizing $\int_\Xi \big[c(x;\xi) + h(x;\xi)\big]\mu(\mathrm{d}\xi)$ subject to $x \in X_0 \subset \mathbb{R}^n$, where X_0 is compact, functions c and f are uniformly continuous on $\mathbb{R}^n \times \mathbb{R}^s$, and functions $c(x;\cdot)$, $h(x;\cdot)$ are Lipschitz for all $x \in X_0$ with constants L_c, L_h; let denote $L := L_c + L_h$. Then ψ is (Berge) u. s. c. at μ w. r. t. $(\mathcal{P}_1(\Xi), W_1)$ and for any $\nu \in \mathcal{P}_1(\Xi)$ we have that $\psi(\nu) \neq \emptyset$ and*

$$|\varphi(\mu) - \varphi(\nu)| \leq L\, W_1(\mu,\nu)$$

The important assumption of the theorem is the Lipschitz continuity of the second stage function h with respect to the (stochastic) variable ξ. Of course, the program with complete fixed linear recourse with (A1)–(A3) falls into this frame. Other stability results one can find in Kaňková [1994a] and Kaňková [1994b], where the Kolmogorov metrics has been used.

Before stating some of the recent results, we introduce a notion of the *minimal information (m. i.) metrics* (see Rachev, Römisch [2000], Dupačová, Gröve-Kuska, Römisch [2000] and references therein). Consider the (general) problem minimize

$$\int_\Xi f_0(x;\xi)\mu(\mathrm{d}\xi) \quad \text{subject to} \quad x \in X_0, \int_\Xi f_j(x;\xi)\mu(\mathrm{d}\xi) \leq 0, j = 1,\dots,d, \tag{3}$$

where $X_0 \subset \mathbb{R}^n$ and $\Xi \subset \mathbb{R}^s$ are closed and $f_j : \mathbb{R}^n \times \Xi \to \overline{\mathbb{R}}$ are normal integrands for $j = 0,\dots,d$. Denote $M(\mu) := \{x \in X_0 : \int_\Xi f_j(x;\xi)\mu(\mathrm{d}\xi) \leq 0,\ j = 1,\dots,d\}$; let $\varphi(\mu)$, $\psi(\mu)$ have the usual meaning. (The problem (1) is a special case of (3).) Let U be a suitable nonempty subset of $\mathbb{R}^n$, let define a class $\mathcal{F}_U := \{f_j(x;\cdot) : x \in X_0 \cap \mathrm{cl}\, U, j = 0,\dots,d\}$, and a set $\mathcal{P}_{\mathcal{F},U}(\Xi) := \Big\{\nu \in \mathcal{P}(\Xi) : \int_\Xi \inf_{\substack{x\in X_0\\ \|x\|\leq K}} f_j(x;\xi)\nu(\mathrm{d}\xi) > -\infty \quad \forall K > 0$ and $\sup_{x\in X\cap \mathrm{cl}\, U} \int_\Xi f_j(x;\xi) < +\infty,\ j = 0,\dots,d\Big\}$. As m. i. metrics we consider a distance of ζ-structure $d_{\mathcal{F},U}(\mu,\nu) := d_{\mathcal{F}_U}(\mu,\nu)$.

Theorem 5 *Let the general assumptions be satisfied, $\mu \in \mathcal{P}_{\mathcal{F},U}(\Xi)$, and assume that $\psi(\mu) \neq \emptyset$, U is an open bounded neighbourhood of $\psi(\mu)$, $x \mapsto \int_\Xi f_0(x;\xi)\mu(\mathrm{d}\xi)$ is Lipschitz continuous on $X \cap \mathrm{cl}\, U$ if $d \geq 1$, and $x \mapsto M_x^{-1}(\mu)$ is metrically regular at each pair $(\bar{x}, 0)$ with $\bar{x} \in \psi(\mu)$. Then $\psi_U : (\mathcal{P}_{\mathcal{F},U}(\Xi), d_{\mathcal{F},U}(\mu,\nu)) \rightrightarrows \mathbb{R}^n$ is (Berge) u. s. c. at μ, and there exist $\delta > 0$, $L > 0$ such that for any $\nu \in \mathcal{P}_{\mathcal{F},U}(\Xi)$ for which $d_{\mathcal{F},U}(\mu,\nu) < \delta$ we have that $\psi_U(\nu)$ is a CLM set w. r. t. U (i. e., $\psi_U(\nu) \subset U$), and*

$$|\varphi(\mu) - \varphi_U(\nu)| \leq L d_{\mathcal{F},U}(\mu,\nu)$$

A definition of the metrical regularity and criteria for it are given in Rockafellar, Wets [1997], Section 9G. The metrics regularity becomes important when the last theorem is applied to the problem with probabilistic constraints. A definition of the CLM (complete local minimizing) set one can find for example in Rachev, Römisch [2000].

The m. i. metrics is hard to work with. Hence, we look for an enlarged class $\mathcal{F}$, and, so, another metrics bounding m. i. metrics above. Such *canonical class* could be formed by locally Lipschitz functions, leading to the Fortet–Mourier metrics presented above. As for any nonempty bounded set $U \subset \mathbb{R}^n$ one has $\mathcal{P}_2(\Xi) \subset \mathcal{P}_{\mathcal{F},U}(\Xi)$, the next theorem from Rachev, Römisch [2000] (theorem 3.3) extends the stability results of Römisch, Schultz [1991b]:

Theorem 6 *Let (A1), (A2) be satisfied, let $\mu \in \mathcal{P}_2(\Xi)$ and let $\psi(\mu) \neq \emptyset$. Then there exist $\delta > 0$ and $L > 0$ such that for any $\nu \in \mathcal{P}_2(\Xi)$ with $\zeta_2(\mu,\nu) < \delta$ we have*

$$|\varphi(\mu) - \varphi(\nu)| \leq L\zeta_2(\mu,\nu).$$

As discussed in their paper, the two metrics W_2 and ζ_2 may have different assymptotic properties, and when (A3) is fulfilled, then the theorem is valid with $\mu \in \mathcal{P}_1(\Xi)$ and with the metrics ζ_1. The last theorem is applied by Dupačová, Gröve-Kuska, Römisch [2000] for a problem of optimal scenario reduction. In Pflug [2001], an example of the optimal scenario tree construction, based on minimizing the Fortet–Mourier metrics of order p, is given. It is shown that it is possible to transform this problem to the case where $p = 1$.

Stability of optimal solution set

Some of the results on (semi-) continuity of optimal solution set have been already given in previous sections. In order to quantify this behaviour, one has to introduce a *growth condition* for the objective function. The notion of the *strong convexity* was explored in many papers, in detail in Römisch, Schultz [1993], and Dentcheva, Römisch, Schultz [1995]. In Shapiro [1994], another type of growth condition is given. Dupačová, Römisch [1998] have adopted another approach via a notion of ε-optimal solutions.

As an example of the result on stability of optimal solution set, we cite theorem 2.7 in Römisch,Schultz [1991b]. Consider the program

$$\min_{x \in X_0} F(x;\mu) := c(x) + \int_\Xi h(x;\xi)\mu(\mathrm{d}\xi)$$

where $X_0 \subset \mathbb{R}^n$ is nonempty convex polyhedron, c is convex quadratic function, h is linear recourse function (with q and T non-stochastic). Denote $Q_\mu(\chi) := \int_\Xi h(x,\xi)\mu(\mathrm{d}\xi)$ where $\chi = Tx$.

Theorem 7 *Fix $\mu \in \mathcal{P}_1(\Xi)$, let $\psi(\mu)$ be nonempty bounded and let $Q_\mu(\cdot)$ be strongly convex on an open convex set V containing $T(\psi(\mu))$. Then there exist $\delta > 0$ and $L > 0$ such that for any $\nu \in \mathcal{P}_1(\Xi)$ with $W_1(\mu,\nu) < \delta$ we have*

$$\Delta_{\mathrm{H}}(\psi(\mu), \psi(\nu)) \leq L \cdot W_1(\mu,\nu)^{1/2},$$

where $\Delta_{\mathrm{H}}(A_1, A_2)$ is the Hausdorff distance of the sets A_1, A_2 (for a definition see e. g. Rockafellar, Wets [1997]). The exponent $1/2$ is stated as optimal, again. More sophisticated conditions and theory are given in new papers of Rachev, Römisch [2000], Dupačová, Gröve-Kuska, Römisch [2000], we refer also to the book of Rockafellar, Wets [1997].

Stability of problems with probabilistic constraints

A question concerning stability of problems with probabilistic constraints is more complicated: one have to take care of the stability of feasibility sets $M(\cdot)$. This is the place where a regularity condition (as the one from the Theorem 5) applies. The majority results of this class is tied with the α (pseudo-) metrics and its modifications (see Römisch, Schultz [1991a, b], Henrion [2000], Rachev, Römisch [2000]), and with the Kolmogorov metrics (Shapiro [1994], Kaňková [1997]).

Future plans

In this article we have shown some basic findings from the literature about the stability of stochastic programs with respect to the concept of probability metrics. Taking into account other (not mentioned) results from an already wide existing literature, the author plans on possible extensions, especially in applying the results: stochastic estimates, scenario construction, are possible areas.

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